
Abridged Riemann's Last Theorem

Aric-Behzad-Canaanie

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Abstract

The central idea of this article is to introduce and prove a special form of the zeta function as proof of Riemann's last theorem. The newly proposed zeta function contains two sub functions, namely $f_1(b, s)$ and $f_2(b, s)$. The unique property of $\zeta(s) = f_1(b, s) - f_2(b, s)$ is that as b tends toward ∞ , the equality $\zeta(s) = \zeta(1 - s)$ is transformed into an exponential expression for the zeros of the zeta function. At the limiting point, we simply deduce that the exponential equality is satisfied if and only if $\Re(s) = \frac{1}{2}$. Consequently, we conclude that the zeta function cannot be zero if $\Re(s) \neq \frac{1}{2}$, hence proving Riemann's last theorem.

Introduction

Riemann's last theorem (RSLT) is very well known and providing a traditional introduction is not the most efficient way to achieve the objective of this work. This subject is well explained in countless publications [(1), (4), (13), (14)] and the assumption is that the reader understands this topic; therefore, instead of explaining RSLT, please consider a brief history of a singular number known as zero. This number reformed our understanding of mathematics, and it's responsible for many advancements and achievements in mathematics. For example, the solution for an equation or function is the zero(s) of that equation or function. It's not so hard to see that the entire calculus is the solution to $\frac{0}{0}$. It's mysterious why this number reveals so much information. For instance, by knowing the zeros of any system, we can construct an accurate model of that system. This work is based on the behavior of functions close to infinity that has been rewritten in a more known and familiar methods. And the paragraph below has been kept as a testimony to the greatness of Riemann's work. Where he revealed an important connection between the zeros of the zeta function and the prime numbers.

The English philosopher Adelard of Bath is known to have adopted the numeral zero to the number system in the 12th century [(5)]. This critical introduction has initiated a journey for this number with respect to the roots of mathematical functions. Over the centuries, the number zero has become the most important source of challenges in mathematics. One such challenge involves proving a conjecture, which states that the real part of the nontrivial

Aric-Behzad-Canaanie
Aecom/Nasa Ames Research Center
M/S 213-8 Bldg. 213, Moffett Field, CA 94035
American Mensa
1315 Brookside Dr., Hurst, TX 76053
E-mail: behzad.canaanie@member.mensa.org

zeros of the zeta function [(1), Chap. X] is $1/2$. This conjecture was first published by Georg Friedrich Bernhard Riemann in 1859 [(2), Chap. I] and is important because it provides a solid and irreplaceable foundation to gain information about the distribution of prime numbers [(4)]. In this work, we present proof of this conjecture.

Assumptions

$1 \leq x \in \mathbb{R}; n = \lfloor x \rfloor \in \mathbb{N}; \Gamma(s) \neq 0, s \in \mathbb{C}; \{ \} \neq \{0\} \neq \{0, 0\} \neq \dots$

Boundary

The functions, expressions, and Equations [(4)–(12)] are restricted to the critical region ($0 < \Re(s) < 1$) unless otherwise noted.

Consideration

The nontrivial roots of the zeta function are considered for validating all relevant statements and conclusions.

Definitions

For $s = \sigma + it$; $\sigma, t \in \mathbb{R}; i = \sqrt{-1}$, the zeta function [(1), Chap. I] is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{n=1}^{\infty} \frac{1}{n^{\sigma+it}} \quad (1)$$

This function converges for $\Re(s) > 1$ and meromorphically continues over the entire complex plane with a simple pole residue of 1 at $s = 1$.

The zeta function then satisfies the functional equation shown below [(1), Chap. II]:

$$\zeta(s) = \frac{2^s}{\pi^{1-s}} \sin\left(\frac{\pi}{2}s\right) \Gamma(1-s) \zeta(1-s) \quad (2)$$

Equation (1) can now be rewritten [(1), Chap. II] for the critical region ($0 < \Re(s) = \sigma < 1$) as

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=1}^{\infty} \int_n^{n+1} \left(\frac{1}{n^s} - \frac{1}{x^s} \right) dx = s \left(\frac{1}{s-1} - \int_1^{\infty} \frac{x - \lfloor x \rfloor}{x^{s+1}} dx \right) \quad (3)$$

Riemann's last theorem

The real part of the nontrivial zeros ($\{s \mid 0 < \Re(s) < 1, \zeta(s) = 0\}$) of the zeta function is $1/2$.

Proof

By applying the summation and integration properties to the left-hand-side function of (3), we can say that

$$\begin{aligned}
\zeta(s) &= \frac{1}{s-1} + \sum_{n=1}^{\infty} \int_n^{n+1} \left(\frac{1}{n^s} - \frac{1}{x^s} \right) dx \\
&= \frac{1}{s-1} + \sum_{n=1}^{\infty} \int_n^{n+1} \left(\frac{1}{n^s} \right) dx - \sum_{n=1}^{\infty} \int_n^{n+1} \left(\frac{1}{x^s} \right) dx \\
&= \frac{-1}{1-s} + \left(\sum_{n=1}^{\infty} \left(\frac{1}{n^s} \right) - \int_1^{\infty} \frac{1}{x^s} dx \right) \\
&= \frac{-1}{1-s} + \lim_{b \rightarrow \infty} \left(\sum_{n=1}^b \left(\frac{1}{n^s} \right) - \frac{b^{1-s}}{1-s} + \frac{1}{1-s} \right)
\end{aligned}$$

Therefore, we obtain a new zeta function (Transcendental Zeta Function, $0 < \Re(s) \& s \neq 1$)

$$\zeta(s) = \lim_{b \rightarrow \infty} \left(\sum_{n=1}^b \left(\frac{1}{n^s} \right) - \frac{b^{1-s}}{1-s} \right) \quad (4)$$

Further, we can generalize the function on the far right of (3) as follows:

$$\begin{aligned}
\zeta(s) &= \frac{s}{s-1} - s \int_1^{\infty} \frac{x - [x]}{x^{s+1}} dx \\
&= \frac{s}{s-1} - \sum_{n=1}^{\infty} \left(s \int_n^{n+1} \frac{x-n}{x^{s+1}} dx \right) \\
&= \frac{s}{s-1} - \sum_{n=1}^{b-1} \left(s \int_n^{n+1} \frac{x-n}{x^{s+1}} dx \right) - \underbrace{s \int_b^{\infty} \frac{x-[x]}{x^{s+1}} dx}_{O_1} \\
&= \frac{s}{s-1} - s \int_1^b \frac{x}{x^{s+1}} dx - s \sum_{n=1}^{b-1} \left(\int_n^{n+1} \frac{-n}{x^{s+1}} dx \right) - O_1 \\
&= \frac{-s}{1-s} + s \int_b^1 \frac{1}{x^s} dx - s \sum_{n=1}^{b-1} n \left(\int_{n+1}^n \frac{1}{x^{s+1}} dx \right) - O_1 \\
&= \frac{-s}{1-s} + \frac{s}{1-s} - \frac{s}{1-s} b^{1-s} + \sum_{n=1}^{b-1} \left(\frac{n}{n^s} - \frac{n}{(n+1)^s} \right) - O_1
\end{aligned}$$

Considering $\frac{-s}{1-s} b^{1-s} = \frac{b}{b^s} - \frac{b^{1-s}}{1-s}$, we have

$$\begin{aligned}
\zeta(s) &= \frac{b}{b^s} - \frac{b^{1-s}}{1-s} + \left(\frac{1}{1^s} - \frac{1}{2^s} + \frac{2}{2^s} - \frac{2}{3^s} + \dots + \frac{b-1}{(b-1)^s} - \frac{b-1}{b^s} \right) - O_1 \\
&= \frac{b}{b^s} - \frac{b^{1-s}}{1-s} + \sum_{n=1}^{b-1} \left(\frac{1}{n^s} \right) - \frac{b}{b^s} + \frac{1}{b^s} - \underbrace{s \int_b^{\infty} \frac{x-[x]}{x^{s+1}} dx}_{O_1}
\end{aligned}$$

Thus, we obtain a new zeta function (valid for $b = 1, 2, 3, \dots$) form of the right-hand side (RHS) function of (3) as (ABC Zeta Function, $0 < \Re(s) \& s \neq 1$)

$$\zeta(s) = \sum_{n=1}^b \left(\frac{1}{n^s} \right) - \frac{b^{1-s}}{1-s} - s \int_b^{\infty} \frac{x-[x]}{x^{s+1}} dx, \quad b \in \mathbb{N} \quad (5)$$

Note that $b = 1$ provides the RHS function of (3). By considering the limit of (5) as b tends toward infinity, (4) can be proved by an alternative method.

We can substitute $(1 - s)$ for (s) and considering (4), and (5), we can derive that(substituted c for b for the ABC zeta function for clarity)

$$\begin{aligned} \lim_{b \rightarrow \infty} \left(\sum_{n=1}^b \left(\frac{1}{n^s} \right) - \frac{b^{1-s}}{1-s} \right) &= \sum_{n=1}^c \left(\frac{1}{n^s} \right) - \frac{c^{1-s}}{1-s} - s \int_c^{\infty} \frac{x - [x]}{x^{s+1}} dx, \quad c \in \mathbb{N} \\ \lim_{b \rightarrow \infty} \left(\sum_{n=1}^b \left(\frac{1}{n^{1-s}} \right) - \frac{b^s}{s} \right) &= \sum_{n=1}^c \left(\frac{1}{n^{1-s}} \right) - \frac{c^s}{s} - (1-s) \int_c^{\infty} \frac{x - [x]}{x^{2-s}} dx, \quad c \in \mathbb{N} \end{aligned} \quad (6)$$

Let ρ be the zeros of the zeta function. Considering (2) and the symmetry of the zeta function, we can say that if $\zeta(\rho) = 0$, then $\overline{\zeta(1-\rho)}$ will also be equal to zero. Therefore, we may assume that the roots of the zeta functions would occur at $\zeta(\rho) = \overline{\zeta(1-\rho)}$. Consequently considering the RHS functions of (6), for all $b \in \mathbb{N}$ we have

$$\begin{aligned} \zeta(\rho) = \overline{\zeta(1-\rho)} &\Rightarrow \sum_{n=1}^b \left(\frac{1}{n^\rho} \right) - \frac{b^{1-\rho}}{1-\rho} - \rho \int_b^{\infty} \frac{x - [x]}{x^{\rho+1}} dx \\ &= \sum_{n=1}^b \left(\frac{1}{n^{1-\rho}} \right) - \overline{\left(\frac{b^\rho}{\rho} \right)} - (1-\rho) \int_b^{\infty} \frac{x - [x]}{x^{2-\rho}} dx \end{aligned} \quad (7)$$

Moving the subfunctions $\sum_{n=1}^b \left(\frac{1}{n^{1-\rho}} \right)$ to the RHS and $-\frac{b^{1-\rho}}{1-\rho}$ to the LHS gives

$$\begin{aligned} \zeta(\rho) = \overline{\zeta(1-\rho)} &\Rightarrow \sum_{n=1}^b \left(\frac{1}{n^\rho} \right) - \sum_{n=1}^b \left(\frac{1}{n^{1-\rho}} \right) - \rho \int_b^{\infty} \frac{x - [x]}{x^{\rho+1}} dx \\ &= \frac{b^{1-\rho}}{1-\rho} - \overline{\left(\frac{b^\rho}{\rho} \right)} - (1-\rho) \int_b^{\infty} \frac{x - [x]}{x^{2-\rho}} dx \end{aligned} \quad (8)$$

Important note all terms of (8) are finite for any arbitrary large b . In the following steps, we will take limit $b \rightarrow \infty$ for simplicity. However, keep in mind that we could keep all terms of the ABC zeta function finite and perform algebraic operations as needed and then take limit $b \rightarrow \infty$ to obtain the terms of the Transcendental zeta function.

Taking the limit of (8) as $b \rightarrow \infty$ gives

$$\begin{aligned} \zeta(\rho) = 0 &\Rightarrow \zeta(\rho) = \overline{\zeta(1-\rho)} \\ &\Rightarrow \sum_{n=1}^{\infty} \left(\frac{1}{n^\rho} \right) - \sum_{n=1}^{\infty} \overline{\left(\frac{1}{n^{1-\rho}} \right)} = \lim_{b \rightarrow \infty} \left(\frac{b^{1-\rho}}{1-\rho} - \overline{\left(\frac{b^\rho}{\rho} \right)} \right) \end{aligned} \quad (9)$$

Note that RHS is zero if $\Re(\rho) = 1/2$.

We know that Riemann encountered function (1), which does not converge to a finite value in the critical strip. He didn't stop there and showed that we are able to convert that function to a convergence function in the critical strip(3). For the rest of this proof, it is essential to understand that Riemann's work is correct and divergence of a function is just the beginning of the analytic continuation. For example, we know that the Euler product $(\prod_p (1 - p^{-s})^{-1})$ [(1), Chap. I] is a divergence function in the critical strip. However it's equal to function (1) and consequently (3) in the critical. This is the connection that enables us to connect the prime numbers to the zeros of the zeta function in the critical strip. This unique connection(one-to-one correspondence) gave us the essential foundation to gain information about prime

numbers. Please keep in mind that one-to-one correspondence is the crucial property of analytic continuation. It is irrational to assume that establishing a mathematical relation between the Euler product and analytic continuation of the Riemann zeta function in the critical region is wrong because the Euler product does not exist (diverges $\sigma > 1$ [(1), Chap. I]).

Moreover, we have proved the transcendental zeta function (4) with two different methods. The transcendental zeta function is a zeta function that is made from two divergence sub-functions. The fascinating part of the transcendental zeta function is the difference between two infinite values giving a finite value. This shows having convergence terms (sub-functions) is not a requirement for the convergence of a function. To put it differently, we see that function (1) minus $\lim_{b \rightarrow \infty} \left(\frac{b^{1-s}}{1-s} \right)$ is well-behaved and regular in the critical strip. The analytic continuation of $\sum_{n=1}^{\infty} \left(\frac{1}{n^p} \right)$ is $\zeta(\rho)$, and because $\zeta(\rho) = \overline{\zeta(1-\rho)}$, we conclude there is no difference between $\sum_{n=1}^{\infty} \left(\frac{1}{n^p} \right)$ and $\sum_{n=1}^{\infty} \left(\frac{1}{n^{1-\rho}} \right)$. In other words, if $\zeta(\rho) = 0$, we can say $\zeta(\rho) - \overline{\zeta(1-\rho)} = 0$ therefore $\sum_{n=1}^{\infty} \left(\frac{1}{n^p} \right) - \sum_{n=1}^{\infty} \left(\frac{1}{n^{1-\rho}} \right)$ is equal to zero despite the divergence of the sub-functions $\sum_{n=1}^{\infty} \left(\frac{1}{n^p} \right)$ and $\sum_{n=1}^{\infty} \left(\frac{1}{n^{1-\rho}} \right)$. (You can find further details and comprehensive proofs for this argument in the Riemann's last theorem article [(18), pages 5-7].

Lastly, consider $\zeta(s) = \frac{1}{1-2^{1-s}} \sum_{n=1}^{\infty} \frac{-1^{n-1}}{n^s}$ [(1), Chap. II] which we can rewrite as $\zeta(s) = \left(\sum_{n=1}^{\infty} \frac{-1^{n-1}}{n^s} \right) \left(1 - \frac{2}{2^s} \right)^{-1}$. For simplicity and clarity, let $\alpha = \left(1 - \frac{2}{2^s} \right)$ and $\beta = \left(1 - \frac{2}{2^{1-s}} \right)$. Thus, we can say

$$\begin{aligned}
& \zeta(\rho) - \overline{\zeta(1-\rho)} = 0 \\
& \Leftrightarrow \left(\sum_{n=1}^{\infty} \frac{-1^{n-1}}{n^{\rho}} \right) (\alpha)^{-1} - \overline{\left(\sum_{n=1}^{\infty} \frac{-1^{n-1}}{n^{1-\rho}} \right) (\beta)^{-1}} = 0 \\
& \Leftrightarrow \sum_{n=1}^{\infty} \left(\left(\frac{-1^{n-1}}{n^{\rho}} \right) (\alpha)^{-1} - \overline{\left(\frac{-1^{n-1}}{n^{1-\rho}} \right) (\beta)^{-1}} \right) = 0 \tag{10} \\
& \Leftrightarrow \sum_{n=1}^{\infty} \left(\left(\frac{1}{n^{\rho}} - \frac{2}{(2n)^{\rho}} \right) (\alpha)^{-1} - \overline{\left(\frac{1}{n^{1-\rho}} - \frac{2}{(2n)^{1-\rho}} \right) (\beta)^{-1}} \right) = 0 \\
& \Leftrightarrow \sum_{n=1}^{\infty} \left(\frac{1}{n^{\rho}} \alpha (\alpha)^{-1} - \overline{\frac{1}{n^{1-\rho}} \beta (\beta)^{-1}} \right) = 0
\end{aligned}$$

Consequently, since $\alpha(\alpha)^{-1} = 1 = \overline{\beta(\beta)^{-1}}$ clearly, we have

$$\zeta(\rho) - \overline{\zeta(1-\rho)} = 0 \Leftrightarrow \sum_{n=1}^{\infty} \left(\frac{1}{n^{\rho}} \right) - \sum_{n=1}^{\infty} \overline{\left(\frac{1}{n^{1-\rho}} \right)} = 0 \tag{11}$$

Now consider (9), we see if either ρ or $1-\rho$ has real part bigger than $1/2$, then one of RHS terms of (9) is bigger than the other and the expression blows up. Consequently considering (11) we can say LHS of (9) cannot be zero if the $\Re(\rho)$ and $\Re(1-\rho)$ are not equal to $1/2$. In other words $\Re(\rho) \neq \frac{1}{2} \Rightarrow \zeta(\rho) \neq \overline{\zeta(1-\rho)}$. Hence, we conclude that

$$\sigma \neq \frac{1}{2} \Leftrightarrow \zeta(s) \neq 0 \tag{12}$$

This completes the proof that the real part of the nontrivial zeros of the zeta function is equal to $1/2$.

Over the years, hundreds of mathematical theories have been built upon the assumption that Riemann's last theorem is true. Therefore, considerable efforts have been made by several of the best mathematical minds around the world to protect the legitimacy of these theories. However, in this work we have finally proved this famous theorem that had resisted all efforts to be proven for over one and a half centuries.



Declarations

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Conflict of interest

The author declares that he has no affiliations with or involvement in any organization or entity with any financial interest (such as honoraria, educational grants, participation in speakers' bureaus, membership, employment, consultancies, stock ownership, or other equity interest, and expert testimony or patent-licensing arrangements), or non-financial interest (such as personal or professional relationships, affiliations, knowledge or beliefs) in the subject matter or materials discussed in this manuscript.

Availability of data and material

All the data and materials are available upon request.

Code availability

All the code are available upon request.

Author contributions

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References

1. Titchmarsh, E.C., Wolffe, A.: *The Theory of the Riemann Zeta-Function*. Clarendon Press, United Kingdom (1986).
2. Edwards, H.M.: *Riemann's Zeta Function*. Dover Publications, United States (2001).
3. Kadry, S.: *Mathematical Formulas for Industrial and Mechanical Engineering*. Elsevier Science, India (2014).
4. Dimitris, K.: *The Distribution of Prime Numbers*. American Mathematical Society, United States (2020).
5. Cochrane, L.: *Adelard of Bath: The First English Scientist*. British Museum Press, London (1994).

6. Berndt, B.C.: *Ramanujan's Notebooks: Part I*. Springer, New York (2012).
7. Berndt, B.C.: *Ramanujan's Notebooks: Part II*. Springer, New York (2012).
8. Berndt, B.C.: *Ramanujan's Notebooks: Part III*. Springer, New York (2012).
9. Berndt, B.C.: *Ramanujan's Notebooks: Part IV*. Springer, New York (2012).
10. Berndt, B.C.: *Ramanujan's Notebooks: Part V*. Springer, New York (2012).
11. Choi, J., Srivastava, H.M.: *Zeta and Q-Zeta Functions and Associated Series and Integrals*. Elsevier Science, Netherlands (2012).
12. Ghusayni, B.: *Classical Real Analysis*. Badih Ghusayni, N.p. (2009).
13. Ghusayni, B.: *Number Theory from an Analytic Point of View*. Badih Ghusayni, N.p. (2004).
14. Hardy, G.H., Wright, E.M., Wiles, A., Silverman, J., Heath-Brown, R.: *An Introduction to the Theory of Numbers*. Oxford University Press, Oxford (2008).
15. Ivic, A.: *The Riemann Zeta-Function: Theory and Applications*. Dover Publications, United States (2012).
16. Polchinski, J.: *String Theory: Volume 1, An Introduction to the Bosonic String*. Cambridge University Press, Cambridge (1998).
17. Voronin, S.M., Koblitz, N., Karatsuba, A.A.: *The Riemann Zeta-Function*. De Gruyter, Germany (2011).
18. Canaanie, A.B.: *Riemann's Last Theorem*, USA arXiv:2201.00615 (2021).