Abridged Riemann's Last Theorem

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Abstract

The central idea of this article is to introduce and prove a special form of the zeta function as proof of Riemann's last theorem. The newly proposed zeta function contains two sub functions, namely $f_1(b,s)$ and $f_2(b,s)$. The unique property of $\zeta(s) = f_1(b,s) - f_2(b,s)$ is that as b tends toward ∞, the equality $\zeta(s) = \zeta(1-s)$ is transformed into an exponential expression for the zeros of the zeta function. At the limiting point, we simply deduce that the exponential equality is satisfied if and only if $\Re(s) = \frac{1}{2}$. Consequently, we conclude that the zeta function cannot be zero if $\Re(s) \neq \frac{1}{2}$, hence proving Riemann's last theorem.

Assumptions

$1 \leq x \in \mathbb{R}$; $n = \lfloor x \rfloor \in \mathbb{N}$; $\Gamma(s) \neq 0$, $s \in \mathbb{C}$; $\{\ \} \neq \{0\} \neq \{0, 0\} \neq \ldots$.

Boundary

The functions, expressions, and Equations [(4)–(14)] are restricted to the critical region $(0 < \Re(s) < 1)$ unless otherwise noted.

Consideration

The nontrivial roots of the zeta function are considered for validating all relevant statements and conclusions.

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Definitions

For s =σ+it ; σ, t ∈ ℝ; i = $\sqrt{-1}$, the zeta function [*(1), Chap. I*] is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{n=1}^{\infty} \frac{1}{n^{\sigma+it}} \tag{1}$$

This function converges for $\Re(s) > 1$ and meromorphically continues over the entire complex plane with a simple pole residue of 1 at s = 1.

The zeta function then satisfies the functional equation shown below [*(1), Chap. II*]:

$$\zeta(s) = \frac{2^s}{\pi^{1-s}} sin\left(\frac{\pi}{2}s\right)\Gamma(1-s)\zeta(1-s) \tag{2}$$

Equation (1) can now be rewritten [*(1), Chap. II*] for the critical region ($0 < \Re(s) = \sigma < 1$) as

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=1}^{\infty} \int_n^{n+1} \left(\frac{1}{n^s} - \frac{1}{x^s}\right) dx = s\left(\frac{1}{s-1} - \int_1^{\infty} \frac{x - \lfloor x \rfloor}{x^{s+1}} dx\right) \tag{3}$$

Riemann's last theorem

The real part of the nontrivial zeros ($\{ s \mid 0 < \Re(s) < 1,\ \zeta(s) = 0 \}$) of the zeta function is 1/2.

Proof

By applying the summation and integration properties to the left-hand-side function of (3), we can say that

$$\begin{aligned}
\zeta(s) &= \frac{1}{s-1} + \sum_{n=1}^{\infty} \int_n^{n+1} \left(\frac{1}{n^s} - \frac{1}{x^s}\right) dx \\
&= \frac{1}{s-1} + \sum_{n=1}^{\infty} \int_n^{n+1} \left(\frac{1}{n^s}\right) dx - \sum_{n=1}^{\infty} \int_n^{n+1} \left(\frac{1}{x^s}\right) dx \\
&= \frac{-1}{1-s} + \left(\sum_{n=1}^{\infty} \left(\frac{1}{n^s}\right) - \int_1^{\infty} \frac{1}{x^s} dx\right) \\
&= \frac{-1}{1-s} + \lim_{b\to\infty} \left(\sum_{n=1}^{b} \left(\frac{1}{n^s}\right) - \frac{b^{1-s}}{1-s} + \frac{1}{1-s}\right)
\end{aligned}$$

Therefore, we obtain a new zeta function (Transcendental Zeta Function, $0 < \Re(s) \& s \neq 1$)

$$\zeta(s) = \lim_{b\to\infty} \left(\sum_{n=1}^{b} \left(\frac{1}{n^s}\right) - \frac{b^{1-s}}{1-s}\right) \tag{4}$$

Notice that the transcendental zeta function is a zeta function that is made from two divergence sub-functions. The fascinating part of the transcendental zeta function is the difference between two infinite values giving a finite value. This shows having convergence terms (sub-functions) is not a requirement for the convergence of a function. To put it differently, we see that function (1) minus $\lim_{b\to\infty}\left(\frac{b^{1-s}}{1-s}\right)$ is well-behaved and regular in the critical strip.

Further, we can generalize the function on the far right of (3) as follows:

$$
\begin{aligned}
\zeta(s) &= \frac{s}{s-1} - s\int_1^\infty \frac{x-\lfloor x\rfloor}{x^{s+1}}dx \\
&= \frac{s}{s-1} - \sum_{n=1}^{\infty}\left(s\int_n^{n+1} \frac{x-n}{x^{s+1}}dx\right) \\
&= \frac{s}{s-1} - \sum_{n=1}^{b-1}\left(s\int_n^{n+1} \frac{x-n}{x^{s+1}}dx\right) - \underbrace{s\int_b^\infty \frac{x-\lfloor x\rfloor}{x^{s+1}}dx}_{O_1} \\
&= \frac{s}{s-1} - s\int_1^b \frac{x}{x^{s+1}}dx - s\sum_{n=1}^{b-1}\left(\int_n^{n+1} \frac{-n}{x^{s+1}}dx\right) - O_1 \\
&= \frac{-s}{1-s} + s\int_b^1 \frac{1}{x^s}dx - s\sum_{n=1}^{b-1} n\left(\int_{n+1}^{n} \frac{1}{x^{s+1}}dx\right) - O_1 \\
&= \frac{-s}{1-s} + \frac{s}{1-s} - \frac{s}{1-s}b^{1-s} + \sum_{n=1}^{b-1}\left(\frac{n}{n^s} - \frac{n}{(n+1)^s}\right) - O_1
\end{aligned}
$$

Considering $\frac{-s}{1-s}b^{1-s} = \frac{b}{b^s} - \frac{b^{1-s}}{1-s}$, we have

$$
\begin{aligned}
\zeta(s) &= \frac{b}{b^s} - \frac{b^{1-s}}{1-s} + \left(\frac{1}{1^s} - \frac{1}{2^s} + \frac{2}{2^s} - \frac{2}{3^s} + \cdots + \frac{b-1}{(b-1)^s} - \frac{b-1}{b^s}\right) - O_1 \\
&= \frac{b}{b^s} - \frac{b^{1-s}}{1-s} + \sum_{n=1}^{b-1}\left(\frac{1}{n^s}\right) - \frac{b}{b^s} + \frac{1}{b^s} - \underbrace{s\int_b^\infty \frac{x-\lfloor x\rfloor}{x^{s+1}}dx}_{O_1}
\end{aligned}
$$

Thus, we obtain a new zeta function (valid for b = 1, 2, 3...) form of the right-hand side (RHS) function of (3) as (ABC Zeta Function, $0 < \Re(s) \& s \neq 1$)

$$
\zeta(s) = \sum_{n=1}^{b}\left(\frac{1}{n^s}\right) - \frac{b^{1-s}}{1-s} - s\int_b^\infty \frac{x-\lfloor x\rfloor}{x^{s+1}}dx\,,\ b \in \mathbb{N} \tag{5}
$$

Note that b = 1 provides the RHS function of (3). By considering the limit of (5) as b tends toward infinity, (4) can be proved by an alternative method.

We can substitute $(1-s)$ for (s) and considering (4), and (5), we can derive that(

substituted c for b for the ABC zeta function for clarity)

$$
\begin{aligned}
\lim_{b\to\infty}\left(\sum_{n=1}^{b}\left(\frac{1}{n^s}\right) - \frac{b^{1-s}}{1-s}\right) &= \sum_{n=1}^{c}\left(\frac{1}{n^s}\right) - \frac{c^{1-s}}{1-s} - s\int_c^\infty \frac{x-\lfloor x\rfloor}{x^{s+1}}dx\,,\ c \in \mathbb{N} \\
\lim_{b\to\infty}\left(\sum_{n=1}^{b}\left(\frac{1}{n^{1-s}}\right) - \frac{b^s}{s}\right) &= \sum_{n=1}^{c}\left(\frac{1}{n^{1-s}}\right) - \frac{c^s}{s} - (1-s)\int_c^\infty \frac{x-\lfloor x\rfloor}{x^{2-s}}dx\,,\ c \in \mathbb{N}
\end{aligned} \tag{6}
$$

Let ρ be the zeros of the zeta function. Considering (2) and the symmetry of the zeta function, we can say that if $\zeta(\rho) = 0$, then $\overline{\zeta(1-\rho)}$ will also be equal to zero. Therefore, we may assume that the roots of the zeta functions would occur at $\zeta(\rho) = \overline{\zeta(1-\rho)}$. Consequently considering the RHS functions of (6), for all $b \in \mathbb{N}$ we have

$$\begin{aligned}\zeta(\rho) = \overline{\zeta(1-\rho)} \Rightarrow & \sum_{n=1}^{b}\left(\frac{1}{n^{\rho}}\right) - \frac{b^{1-\rho}}{1-\rho} - \rho\int_{b}^{\infty}\frac{x-\lfloor x\rfloor}{x^{\rho+1}}dx \\ &= \sum_{n=1}^{b}\overline{\left(\frac{1}{n^{1-\rho}}\right)} - \overline{\left(\frac{b^{\rho}}{\rho}\right)} - \overline{(1-\rho)\int_{b}^{\infty}\frac{x-\lfloor x\rfloor}{x^{2-\rho}}dx}\end{aligned} \tag{7}$$

Moving the subfunctions $\sum_{n=1}^{b}\overline{\left(\frac{1}{n^{1-\rho}}\right)}$ to the RHS and $-\frac{b^{1-\rho}}{1-\rho}$ to the LHS gives

$$\begin{aligned}\zeta(\rho) = \overline{\zeta(1-\rho)} \Rightarrow & \sum_{n=1}^{b}\left(\frac{1}{n^{\rho}}\right) - \sum_{n=1}^{b}\overline{\left(\frac{1}{n^{1-\rho}}\right)} - \rho\int_{b}^{\infty}\frac{x-\lfloor x\rfloor}{x^{\rho+1}}dx \\ &= \frac{b^{1-\rho}}{1-\rho} - \overline{\left(\frac{b^{\rho}}{\rho}\right)} - \overline{(1-\rho)\int_{b}^{\infty}\frac{x-\lfloor x\rfloor}{x^{2-\rho}}dx}\end{aligned} \tag{8}$$

We know that Riemann encountered function (1), which does not converge to a finite value in the critical strip. He didn't stop there and showed that we are able to convert that function to a convergence function in the critical strip(3). For the rest of this proof, it is essential to understand that Riemann's work is correct and divergence of power series is just the beginning of the analytic continuation. For example, we know that the Euler product ($\prod_{p}(1-p^{-s})^{-1}$) [*(1), Chap. I*] is a divergence function in the critical strip. However it's equal to function (1) and consequently (3) in the critical. This is the connection that enables us to connect the prime numbers to the zeros of the zeta function in the critical strip. This unique connection(one-to-one correspondence) gave us the essential foundation to gain information about prime numbers. It is irrational to assume that establishing a mathematical relation between the Euler product and analytic continuation of the Riemann zeta function in the critical region is wrong because the Euler product does not exist(diverges $\sigma >= 1$ [*(1), Chap. I*]).

Notice that all terms of (8) are finite for any arbitrary large b. In the following steps, we will take limit b→∞ for simplicity. However, keep in mind that we could keep all terms of the ABC zeta function finite and perform algebraic operations as needed and then take limit b→∞ to obtain the terms of the Transcendental zeta function.

Taking the limit of (8) as b→∞ gives

$$\begin{aligned}\zeta(\rho) = 0 &\Rightarrow \zeta(\rho) = \overline{\zeta(1-\rho)} \\ &\Rightarrow \sum_{n=1}^{\infty}\left(\frac{1}{n^{\rho}}\right) - \sum_{n=1}^{\infty}\overline{\left(\frac{1}{n^{1-\rho}}\right)} = \lim_{b\to\infty}\left(\frac{b^{1-\rho}}{1-\rho} - \overline{\left(\frac{b^{\rho}}{\rho}\right)}\right)\end{aligned} \tag{9}$$

Now we want to use the Identity Theorem [*(18), page 125*] to show that LHS of (9) is zero for the zeros of the zeta function .

Consider two analytic functions f and g as follow

$$\begin{aligned}f &= \lim_{b\to\infty}\left(\sum_{n=1}^{b}\left(\frac{1}{n^{z}}\right) - \frac{b^{1-z}}{1-z} - \sum_{n=1}^{b}\overline{\left(\frac{1}{n^{1-z}}\right)} + \overline{\frac{b^{z}}{z}}\right),\ z \in \mathbb{D}_1 := \{s \in \mathbb{C} \mid 0 < \Re(s) < 1\} \\ g &= \lim_{b\to\infty}\left(\sum_{n=1}^{b}\left(\frac{1}{n^{z}}\right) - \sum_{n=1}^{b}\overline{\left(\frac{1}{n^{1-z}}\right)}\right),\ z \in \mathbb{D} := \left\{s \in \mathbb{C} \,\middle|\, \Re(s) = \frac{1}{2}\right\}\end{aligned} \tag{10}$$

Let f, g : $\mathbb{D} \to \mathbb{D}_1$

The "coincidence set"

$$\{z \in \mathbb{D} \mid f = g\} \tag{11}$$

has an accumulation point in $\mathbb{D}$. Therefore considering the Identity Theorem we can say f=g on the domain $\mathbb{D}_1$. In other words

$$\zeta(z) - \overline{\zeta(1-z)} = \sum_{n=1}^{\infty}\left(\frac{1}{n^z}\right) - \sum_{n=1}^{\infty}\overline{\left(\frac{1}{n^{1-z}}\right)},\ z \in \mathbb{D}_1 := \{s \in \mathbb{C} \mid 0 < \Re(s) < 1\} \tag{12}$$

Consequently, for the zeros of the zeta function we have

$$\zeta(\rho) - \overline{\zeta(1-\rho)} = 0 \Leftrightarrow \sum_{n=1}^{\infty}\left(\frac{1}{n^\rho}\right) - \sum_{n=1}^{\infty}\overline{\left(\frac{1}{n^{1-\rho}}\right)} = 0 \tag{13}$$

You can find comprehensive proofs for (13) in the Riemann's last theorem article [*(19), pages 5-7*].
Now consider (9), we see if either ρ or $1-\rho$ has real part bigger than 1/2, then one of RHS terms of (9) is bigger than the other and the expression blows up. Consequently, considering (13) we can say LHS of (9) cannot be zero if the $\Re(\rho)$ and $\Re(1-\rho)$ are not equal to 1/2. In other words $\Re(\rho)\neq\frac{1}{2}\Rightarrow\zeta(\rho)\neq\overline{\zeta(1-\rho)}$. Hence, we conclude that

$$\sigma \neq \frac{1}{2} \Leftrightarrow \zeta(s) \neq 0 \tag{14}$$

This completes the proof that the real part of the nontrivial zeros of the zeta function is equal to 1/2.

Over the years, hundreds of mathematical theories have been built upon the assumption that Riemann's last theorem is true. Therefore, considerable efforts have been made by several of the best mathematical minds around the world to protect the legitimacy of these theories. However, in this work we have finally proved this famous theorem that had resisted all efforts to be proven for over one and a half centuries.

Declarations

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Conflict of interest

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Availability of data and material

All the data and materials are available upon request.

Code availability

All the code are available upon request.

Author contributions

Not applicable.

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